MIXED CIRCUIT DOMINATION NUMBER

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Abstract- A mixed dominating set $S \subset V \cup E$ of a connected graph $G(V, E)$ is mixed circuit dominating set of G if the elements of S along with the ends of the edges in it constitute a circuit. A mixed circuit dominating set S is a minimal mixed circuit dominating set if no proper subset of S is a mixed circuit dominating set. The mixed circuit domination number $\gamma_{mc}(G)$ of G is the minimum cardinality of a mixed circuit dominating set. In this paper we include some basic results on mixed circuit domination, bounds on $\gamma_{mc}(G)$ and its exact values for some standard graphs.

Key Words: mixed circuit domination; mixed circuit domination number

1 Introduction

A set S of vertices in a graph $G(V, E)$ is called a *dominating set* if every vertex $v \in V \setminus S$, is adjacent to an element of S. A dominating set is *minimal dominating set* if no proper subset $S'$of S is a dominating set. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of dominating sets in G. A dominating set with minimum cardinality is called a $\gamma - set$ of G.
It is immediate that every superset of a dominating set of graph G is again a dominating set.
Let $G(V, E)$ be a connected graph. Then $S \subset V$ is called a *connected dominating set* if S is a dominating set and the subgraph induced by S of G is connected. The minimum cardinality of connected dominating sets is called *connected domination number* and is denoted by $\gamma_c(G)$.
For a graph G(V,E) a subset S of $V \cup E$ is a *mixed dominating set* if every element $x \in (V \cup E) \setminus S$ is adjacent or incident to an element of S. In mixed domination, a vertex **v** dominates itself, all vertices adjacent to **v** and all edges incident with **v** and an edge **uv** dominates itself, both vertices **u** and **v** and all edges incident with **u** or **v**. A mixed dominating set is a *minimal mixed dominating set* if no proper subset of S is a mixed dominating set. The *mixed domination number* $\gamma_m(G)$ of G is the minimum cardinality of a mixed dominating set. A $\gamma_m - set$ of a graph G is a mixed dominating set with cardinality $\gamma_m(G)$.
If S is a mixed dominating set and $x \in S$, by a *private neighbour* of x (with respect to S) we mean an element of $V \cup E$ which is dominated by x but not by any other member of S.

2 Mixed Circuit Domination

Throughout this paper, a graph represents a connected graph unless otherwise specified and the darkened elements of $V \cup E$ of graph G indicates the elements of a mixed circuit dominating set.

Definition 2.1. By a graph formed by a subset A of $V \cup E$ we mean a subgraph whose edge set is $A \cap E$ and the vertex set consists the vertices in A together with the ends of the edges in A.

Definition 2.2. A mixed dominating set $S \subset V \cup E$ of a connected graph $G(V, E)$ is mixed circuit dominating set of G if the graph formed by S is a circuit.

ie; $S \subseteq V \cup E$ is a mixed circuit dominating set if

1. each element of $V \cup E \setminus S$ is either adjacent or incident to an element of S and
2. the elements of S together with the ends of the edges in S form a circuit.

Definition 2.3. A mixed circuit dominating set S is said to be a minimal mixed circuit dominating set if no proper subset of S is a mixed circuit dominating set.

Definition 2.4. The mixed circuit domination number $\gamma_{mc}(G)$ of G is the minimum cardinality of a mixed circuit dominating set.

Definition 2.5. A γ_{mc} set of a graph G is a mixed circuit dominating set with cardinality $\gamma_{mc}(G)$.

The mixed circuit domination problem of a graph G is the problem of finding γ_{mc} set of G.

3 Basic Properties

A mixed circuit dominating set S of a graph G is a subset of the vertex set V if and only if $|S| = \gamma_{mc}(G) = 1$. In other words, if $\gamma_{mc}(G) > 1$ then, any mixed circuit dominating set of G must contain an edge.

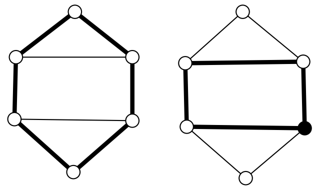

Figure 1: Two minimal mixed circuit dominating sets of a graph are indicated in figure by darkened edges. The second one is a $\gamma_{mc} - set$.

Theorem 1. *Let G be a graph. Suppose that $G \neq K_1$. A mixed circuit dominating set S is minimal if and only if*

1. *every vertex in S has a private neighbor in V and*
2. *every edge e in S has a private neighbor in $(V \cup E) \setminus S$ or the graph formed by $S \setminus \{e\}$ is not a circuit.*

Proof. The sufficient part is straight forward . To prove the necessary part, consider a minimal mixed circuit dominating set S of G. The result is trivial if G is K_1. So let $n \geq 2$. If a vertex $v \in S$ has no private neighbor in V, then any element in the neighbor set of v is dominated by some other element in S. Therefore $|S| > 1$ and since the elements of S form a circuit, S must contain two edges adjacent to v. So the set $S \setminus \{v\}$ will form a mixed circuit dominating set which contradicts the minimality of S.
Now suppose an edge $e \in S$ has no private neighbor in $(V \cup E) \setminus S$ and the graph formed by $S \setminus \{e\}$ is a circuit. Since e has no private neighbor every member of $(V \bigcup E) \setminus S$ is dominated by $S \setminus \{e\}$. Also since S is a mixed circuit dominating set e is dominated by some members of $S \setminus \{e\}$. Thus $S \setminus \{e\}$ forms a mixed circuit dominating set of G, again a contradiction. □

Theorem 2. *For a graph G(V,E), $\gamma_{mc}(G) = 1$ if and only if G is a star.*

Proof. Let G be the star $K_{1,t}, t \geq 0$. Then the vertex v in G of degree t forms a γ_{mc}-set of G. Hence $\gamma_{mc}(G)$=1.
Conversely assume that $\gamma_{mc}(G) = 1$. Let S be a γ_{mc}-set. Since an edge cannot form a circuit, $S = \{v\}$, where $v \in V$. This is true only if v is adjacent to all other vertices in G and all edges in G are incident with v, so that G is a star. □

Remark 3.1. A subset S of $V \cup E$ of cardinality 2 cannot form a circuit. Therefore there exist no graph G with $\gamma_{mc}(G) = 2$.

Theorem 3. *For a graph G, $\gamma_{mc}(G) = 3$ if and only if G is a triangle.*

Proof. If G is a triangle the result is obvious.
Conversely suppose that $\gamma_{mc}(G) = 3$. Then there exist a $\gamma_{mc}-$set S for G with $|S| = 3$. Clearly $S \subset E$. No other subsets of $V \cup E$ of cardinality 3 form a circuit. Since S is a $\gamma_{mc}-$set of cardinality 3 and $S \subset E$, it forms a triangle. Suppose G is not itself a triangle. Then G contains a vertex other than the end vertices of elements of S. Since $S \subset E$ such a vertex cannot be dominated by S, a contradiction.Therefore G is a triangle. □

Theorem 4. *For a graph G(V,E), $\gamma_{mc}(G) = 4$ if and only if G is either C_4 or $G = (K_1 + (rK_1 \cup K_{1,s}) \cup pK_1) + K_1$, where r,s and p are non negative integers not simultaneously zero.*

Proof. If G is C_4 the result is obvious .
Let $G = (K_1 + (rK_1 \cup K_{1,s}) \cup pK_1) + K_1$, where r,s and p be non negative integers such that $r + s + p \geq 0$. There is a vertex in G adjacent to all other vertices in G for any r,s,p. The triangle obtained by putting $r = s = p = 0$ is a subgraph of G for all r,s and p. The edges of this triangle together with the vertex of degree $|G| - 1$ form a mixed circuit dominating set for G. Therefore $\gamma_{mc}(G) \leq 4$. Since G is not a star graph or a triangle by Theorems 2 and 3

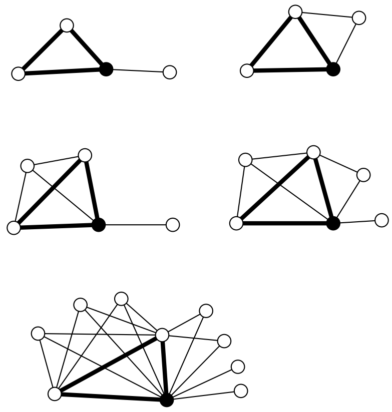

Figure 2: *G*

$\gamma_{mc}(G) \geq 4$. Therefore $\gamma_{mc}(G) = 4$.
Conversely suppose that $\gamma_{mc}(G) = 4$. Then any γ_{mc}-set S must contain either four edges or three edges and a vertex.
If $S \subset E$. Then S forms a C_4 and dominates the edges in this C_4, its four vertices and all edges incident with these vertices. Therefore G is either C_4, $(K_1 + (K_1 \cup K_1)) + K_1$ or K_4
If S contains three edges and a vertex, then the edges in S form a triangle and the vertex in S is a vertex in this triangle. Let S={v,uv,vw,wu}. So all vertices of G must be adjacent to v and all edges must be incident to either u, v or w. Thus G must be one of the graphs mentioned in the statement. □

Theorem 5. *Let G(V,E) be a graph with $|V| = n$. If $\gamma_{mc}(G) < n$ then any γ_{mc}-set of G must contain a vertex.*

Proof. A circuit with k edges can dominate at most k vertices. Therefore any γ_{mc}-set with cardinality less than n must contain a vertex. □

Corollary 6. *If a graph G(V,E) has a γ_{mc}-set $S \subset E$, then $\gamma_{mc}(G) \geq |V|$*

Definition 3.1. A mixed circuit dominating set S is said to visit a vertex if it is an end vertex of some edge in S.

Remark 3.2.
1. A vertex adjacent to a pendant vertex should be in every γ_{mc}-set.
2. A vertex with degree two do not belong to any γ_{mc}-set.
3. A mixed circuit dominating set visits all cut vertices.

Theorem 7. *Let G be a graph with n vertices and m edges. Let S be a γ_{mc}-set for G with $|S| = n$. Then $S \subset E$ if and only if G contains a spanning cycle.*

Proof. An edge dominates only the end vertices of that edge and those edges incident with it. Let $S \subset E$ be a γ_{mc}-set for G. Then every vertex must be an end point of some element of S. The minimal circuit which contains all vertices of G and contains exactly n edges is a spanning cycle of G. Thus elements of S form a spanning cycle if $S \subset E$ and $|S| = n$.
Conversely suppose that G has a spanning cycle. Let S be edge set of this spanning cycle. Then $|S| = n$. Clearly S forms a mixed circuit dominating set. Since $\gamma_{mc}(G)$=n, S is also minimal. □

Corollary 8. *If a graph G(V,E) contains a spanning cycle then $\gamma_{mc}(G) \leq |V|$*

Example 9. *For the graph in figure 3, $\gamma_{mc}(G) < |V|$ though it contains a spanning cycle*

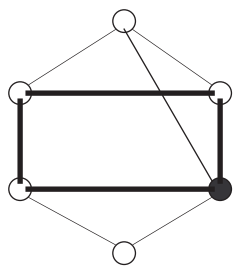

Figure 3: $\gamma_{mc}(G) = 5 < |V|$

Proposition 3.1. *If a graph G has a cut edge other than a pendant edge then G has no mixed circuit dominating set.*

Proof. Let e be a cut edge other than a pendant edge. Then $G \setminus \{e\}$ has two non trivial component say G^* and G^{**}. Since e is a cut edge it belongs to no cycle. Therefore any circuit of G is either a circuit of of G^* or a circuit of G^{**} and any circuit of G^* dominates only edges of G^* and any circuit of G^{**} dominates only edges of G^{**}. Hence the result. □

Remark 3.3. The converse of proposition 3.1 need not be true.The graph G in figure 4 has neither a cut edge nor a mixed circuit dominating set.

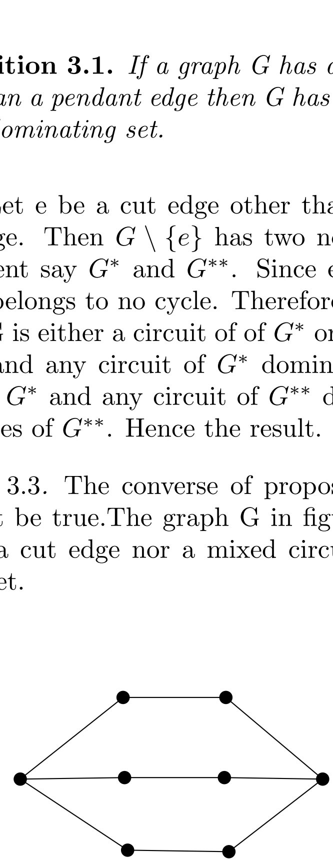

Figure 4: G

Theorem 10. *If an r regular graph G with n vertices has a mixed circuit dominating set then $\gamma_{mc}(G) \geq \lceil \frac{rn}{2(r-1)} \rceil + \lceil \frac{n - \lceil \frac{rn}{2(r-1)} \rceil}{r-2} \rceil$, where $r \geq 3$*

Proof. Let S be a $\gamma_{mc}-$ set of G. Suppose S contains k edges. Assume that S forms a cycle with k edges. Then these k edges dominate all edges and k vertices. Since G is a r regular graph a cycle with k edges can dominate at most (r-1)k edges and since S is a$\gamma_{mc} - set$ $(r-1)k \geq \frac{r}{2}n$. Therefore $k \geq \frac{rn}{2(r-1)}$. ie;$k \geq \lceil \frac{rn}{2(r-1)} \rceil$
And k edges can dominate at most k vertices. A vertex in S can dominate at most r-2 vertices not in that cycle. If $k = \lceil \frac{rn}{2(r-1)} \rceil$ we need at least $\lceil \frac{n - \lceil \frac{rn}{2(r-1)} \rceil}{r-2} \rceil$ to dominate the remaining

$n - \lceil \frac{rn}{2(r-1)} \rceil$ vertices.
Therefore $|S| \geq \lceil \frac{rn}{2(r-1)} \rceil + \lceil \frac{n - \lceil \frac{rn}{2(r-1)} \rceil}{r-2} \rceil$. □

Corollary 11. *If a cubic graph G with n vertices has a mixed circuit dominating set then* $\gamma_{mc}(G) = n$.

Proof. Let S be any $\gamma_{mc} - set$ of G containing k edges. Then these k edges dominate all edges and k vertices of G. Since any circuit in a 3-regular graph is a cycle (otherwise it should contain a vertex of degree greater than or equal to four), any vertex in S dominates exactly one vertex not in S. Therefore S should contain n-k vertices. ie;$|S| = k + (n - k) = n$. □

4 Mixed Circuit Domination Number of Some Standard Graphs

In this section we find the mixed circuit domination number for some standard graphs.

Theorem 12. *The circuit domination number of a complete graph K_n is n, where $n \neq 2$.*

Proof. Let G(V,E) be a complete graph with $|V| = n$. Since G is complete it contains a spanning cycle. Therefore $\gamma_{mc}(G) \leq n$ by Corollary 8.
Let S be any $\gamma_{mc} - set$. Suppose $|S| < n$. Then S contain at least one vertex by Theorem 5. That is S contains at most n-2 edges and n-2 edges can dominate at most n-2 vertices. Thus there exists two vertices say u and v which are not dominated by the edge set of S and these two vertices are not members of the circuit formed by S. Since G is complete there exists an edge between u and v and is not dominated by S. Hence the result. □

Theorem 13. *For m,n $\geq$ 2* $\gamma_{mc}(K_{m,n})$ $= \begin{cases} 2m+1, & \text{if } n > m; \\ 2m, & \text{if } n = m. \end{cases}$

Proof. Let $V = (V_1, V_2)$ be the partition of the vertex set of $K_{m,n}$. Suppose $V_1 = \{u_1, u_2, ..., u_m\}$ and $V_2 = \{v_1, v_2, ..., v_n\}$.
case(i) $n > m \geq 2$
Then $S = \{u_1, u_1v_1, v_1u_2, u_2v_2, ..., u_mv_m, v_mu_1\}$ forms a mixed circuit dominating set. Therefore $\gamma_{mc}(K_{m,n}) \leq 2m + 1$. To get the reverse inequality, consider a mixed circuit dominating set S of $K_{m,n}$ with $|S| = k$. If possible let k$\leq 2m$. Then S contains at least one vertex. That is S contains at most 2m-1 edges. Since there is no odd cycle in $K_{m,n}$ S contains at most 2m-2 edges. That is the edges in S dominates at most 2m-2 vertices of $V_1 \cup V2$. Also since every edge in $K_{m,n}$ has one end in V_1 and the other end in V_2, for each i=1,2 the edges in S can dominate at most m-1 vertices of V_i. Hence there exist two vertices $u_i \in V_1$ and $v_j \in V_2$ which are not dominated by the edges in S. Hence the edge u_iv_j is not dominated by S, a contradiction. Therefore $|S| \geq 2m + 1$. Hence the result.
case(i)$n = m \geq 2$
Then $S = \{u_1v_1, v_1u_2, u_2v_2, ..., u_mv_m, v_mu_1\}$ forms a mixed circuit dominating set. Therefore $\gamma_{mc}(K_{m,m}) \leq 2m$. To get the reverse inequality, consider a mixed circuit dominating set S of $K_{m,m}$ with $|S| = k$. If possible let k$\leq 2m - 1$.Then S contain at least one vertex. That is S contain at most 2m-2 edges. That is the edges in S dominates at most 2m-2 vertices of $V_1 \cup V2$. Also since every edge in $K_{m,m}$ has one end in V_1 and the other end in V_2, for each i=1,2 the edges in S can dominate at most m-1 vertices of V_i. Hence there exist two vertices $u_i \in V_1$ and $v_j \in V_2$ which are not dominated by the edges in S. Hence the edge u_iv_j is not dominated by S, a contradiction. Therefore $|S| \geq 2m$. Hence the result. □

The following lemma will help to determine γ_{mc} of a Wheel graph.

Lemma 14. *Let C be a circuit in the Wheel graph $W_{n+1} = C_n + K_1$ [2] with $|E(C)| \leq n-1$ then C omits three consecutive edges of C_n.*

Proof. Let C be a circuit in $W_{n+1} = C_n + K_1$

with number of edges less than or equal to n-1. Then it must contain the central vertex [since the circuit which does not contain central vertex is C_n and has n edges]
Therefore the circuit C contains at least two edges incident with the central vertex. Let number of edges common to C and C_n be k. If possible suppose C does not omit three consecutive edges of C_n. Then there are n-k edges of C_n which does not belong to the circuit. Therefore the circuit contain at least n-k edges incident with central vertex. Therefore the total number of edges of the circuit C is greater than k+(n-k)=n, which is a contradiction. Hence the result. □

Theorem 15. $\gamma_{mc}(W_{n+1}) = n + 1$

Proof. Consider $W_{n+1} = C_n + K_1$
Then the edges of C_n together with one vertex constitute a mixed circuit dominating set of W_{n+1}.
Therefore $\gamma_{mc}(W_{n+1}) \leq n + 1$
Now let $S \subset V \cup E$ be γ_{mc} set of cardinality less than n+1. Then by Theorem 5 S contains at least one vertex. Thus S contains at most n-1 edges and these edges of S forms a circuit C with $|E(C)| < n - 1$. Therefore by Lemma 14 C omits three consecutive edges of C_n. Thus in this case S cannot dominate the edge in the middle of these three consecutive edges of C_n
Therefore $\gamma_{mc}(W_{n+1}) = n + 1$. □

Corollary 16. *Mixed circuit domination number of Petersen graph (J(5,2,0)) [2] is 10 and any γ_{mc} set contains at least one vertex.*

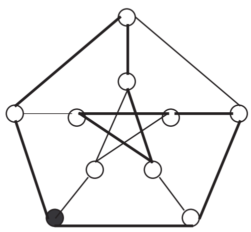

Figure 5: $J(5, 2, 0)$

Proof. Since $J(5, 2, 0)$ is a cubic graph $\gamma_{mc}(J(5, 2, 0)) = 10$ by Corollary 11. Since J(5,2,0) is a non hamiltonian [2] 3- regular graph on 10 vertices and $\gamma_{mc}(J(5, 2, 0)) = 10$, by Theorem 7 any γ_{mc} set contains at least one vertex.

□

Corollary 17. *Mixed circuit domination number of 3 Cube Q_3 [2] is 8*

Proof. Q_3 is a cubic graph with 8 vertices and it has a mixed circuit dominating set, by Corollary $\gamma_{mc}(G) = 8$.

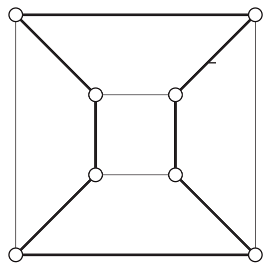

Figure 6: Q_3

□

Corollary 18. *Mixed circuit domination number of 3 Prism [2] is 6 and any γ_{mc} set contains at least one vertex.*

Proof. 3 prism is a cubic graph with 6 vertices and it has a mixed circuit dominating set,by Corollary 11 $\gamma_{mc}(G) = 6$. Since 3 prism is a non hamiltonian 3- regular graph on 6 vertices and $\gamma_{mc}(G) = 6$, by Theorem 7 any γ_{mc} set contains at least one vertex.

□

Corollary 19. *Mixed circuit domination number of Tutte's 8 cage [1] is 30 .*

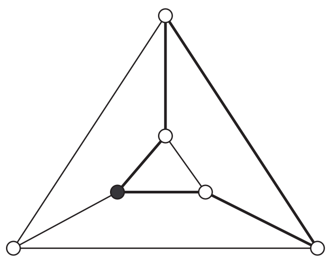

Figure 7: 3 prism

Proof. Tutte's 8 cage is a cubic graph with 30 vertices and it has a mixed circuit dominating set, by Corollary11 $\gamma_{mc}(G) = 30$

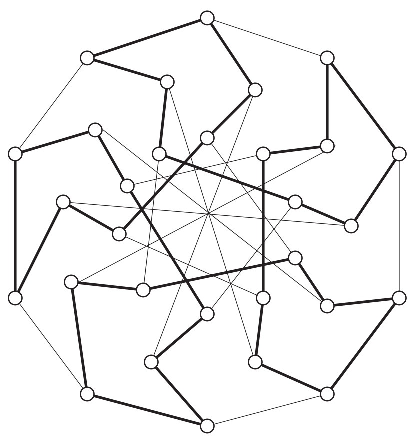

Figure 8: Tutte's 8 cage

□

Corollary 20. *Mixed circuit domination number of Coxecter graph [1] is 28 and any γ_{mc} set contains at least one vertex.*

Proof. Coxecter graph is a cubic graph with 28 vertices and it has a mixed circuit dominating set. Therefore $\gamma_{mc}(G) = 28$ by Corollary 11.

Since it is a 3- regular graph on 8 vertices which has no hamiltonian cycle and $\gamma_{mc}(G) =$

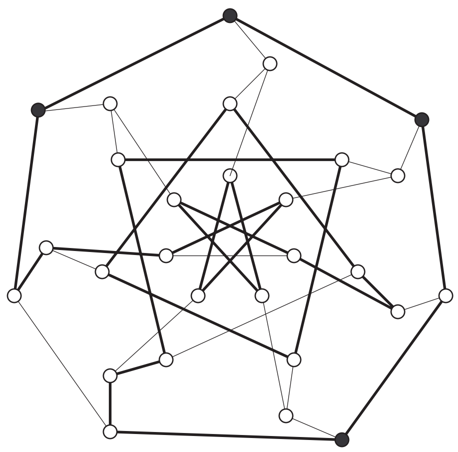

Figure 9: Coexecter

28, by Theorem 7 any γ_{mc} set contains at least one vertex. □

Corollary 21. *Mixed circuit domination number of Heawood graph [2] is 14.*

Proof. Heawood graph is a cubic graph with 14 vertices and it has a mixed circuit dominating set. Therefore $\gamma_{mc}(G) = 14$

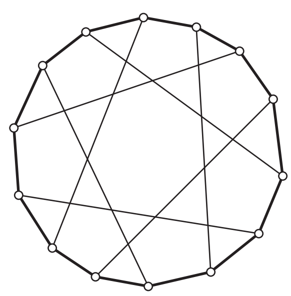

Figure 10: heawood graph

□

Remark 4.1. There are 3- regular graphs which have no mixed circuit dominating set.
For example the graph G obtained by replacing

each vertex of the Petersen graph by a triangle has no mixed circuit dominating set.

Theorem 22. *If G is the graph obtained by replacing each vertex of the Petersen graph by a triangle then G has no mixed circuit dominating set.*

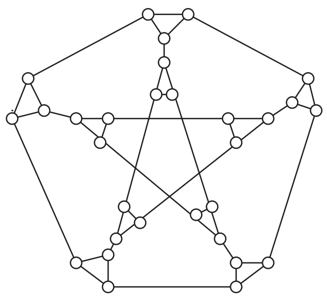

Figure 11: Petersen graph with each vertex replaced by a triangle

Proof. Let G be the graph obtained by replacing each vertex of the Petersen graph by a triangle. Since G is a cubic graph the edges of any mixed circuit dominating set form a cycle. Since Petersen graph $J(5,2,0)$ is not hamiltonian every cycle in $J(5,2,0)$ omits at least one vertex . Therefore every cycle in G omits all the vertices of at least one triangle of G. So any cycle of G cannot dominate the edges of that triangle. Therefore G has no mixed circuit dominating set. $\square$

Another 3-regular graph having no mixed circuit dominating set is given in figure 12

Now again we come back to the computation of mixed circuit domination number of some well known graphs. First consider the Andrasfai graph.
For the additive group $G = Z_{3k-1}, k \geq 1$ and $C \subset Z_{3k-1}$ consisting of elements congruent to 1 modulo 3, the graph Andrasfai graph And(k) is the cayley graph $Cay(G,C)$
For example And(4)=Cay(Z_{11},{1,4,7,10})

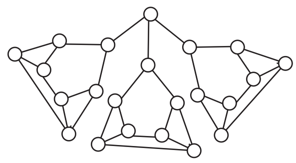

Figure 12: A 3- regular graphs with no mixed circuit dominating set.

Theorem 23. γ_{mc}*(andrasfaigraph And(4))=10*

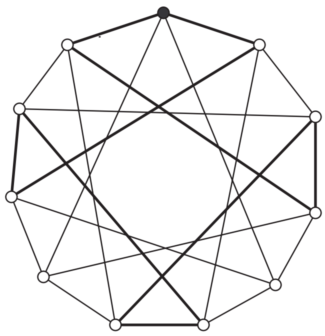

Figure 13: And(4)

Proof. From figure 13 it is clear that And(4) has a mixed circuit dominating set of cardinality 10. Therefore $\gamma_{mc}(And(4)) \leq 10$
Since And(4) is a 4 regular graph $\gamma_{mc}(And(4)) \geq \lceil \frac{4*11}{2*(4-1)} \rceil + \lceil \frac{11-8}{4-2} \rceil = 8 + 2 = 10$. by Theorem 10
Therefore $\gamma_{mc}(And(4)) = 10$ $\square$

Theorem 24. *The mixed circuit domination number of* (Q_4) *[2] is 14.*

Proof. The graph Q_4 in figure 14 is a 4 regular graph of order 16. From the figure it is clear that Q_4 has a mixed circuit dominating set of cardinality 14. Therefore $\gamma_{mc}(Q_4) \leq 14$.
By Theorem 10, $\gamma_{mc}(Q_4) \geq \lceil \frac{4*16}{2*(4-1)} \rceil + \lceil \frac{16-11}{4-2} \rceil = 11 + 3 = 14$
Therefore, $\gamma_{mc}(Q_4) = 14$ $\square$

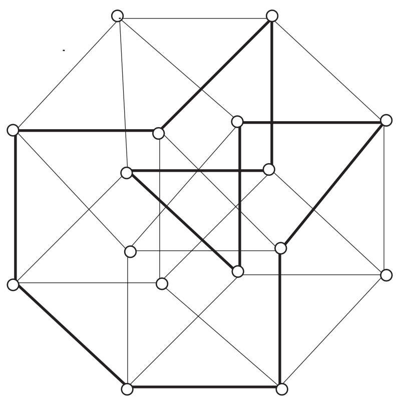

Figure 14: Q_4

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